MATH 214 Chapter 14

Consider a function f such that

$$\begin{aligned} f : \mathcal{D} \subset \mathbb{R}^n &\to \mathbb{R} \\ (x_1, \ldots, x_n) &\to f(x_1, \ldots, x_n) \end{aligned} \tag{1}$$

0.1 Definition 1: Real-valued functions of n variables

A function of n variables is a rule that assigns a unique real number $f(x_1, \ldots, x_n)$ to each point $(x_1, \ldots, x_n)$ in $\mathcal{D}$

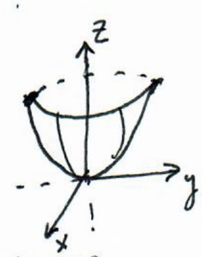
0.2 Definition 2: Domain and Range of functions of n variables

$f(x_1, \ldots, x_n)$ is called *the image of* $(x_1, \ldots, x_n)$ *under* f.

The set $\mathcal{D} \subset \mathbb{R}^n$ is called *the domain of* f.

The set of all real values $f(x_1, \ldots, x_n)$ is called *the Range of* f. It means

$$\text{range of } f = \{f(x_1, \ldots, x_n) : (x_1, \ldots, x_n) \in \mathcal{D}\}$$



Discuss examples of functions of two variables. $z = f(x,y) = x^2 + y^2$

The graph of a real-valued function f of two is the variables is the set of points $(x, y, z) \in \mathbb{R}^3$ such that $z = f(x, y)$ and $(x, y) \in \mathcal{D}$.

0.3 Definition 3: Level Curves for functions of two variables

Assume the graph of a function $z = f(x, y)$ describes a surface S in $\mathbb{R}^3$. For any real value c in the range of f, the intersection of of S with the plane $z = c$ is the curve

$$f(x, y) = c$$

The projection of this curve onto the xy-plane is called a level curve of f. It is also called a *contour curve.*

Exercise: obtain Level curves for

$$z = f(x,y) = \ln(1+x^2+y^2)$$

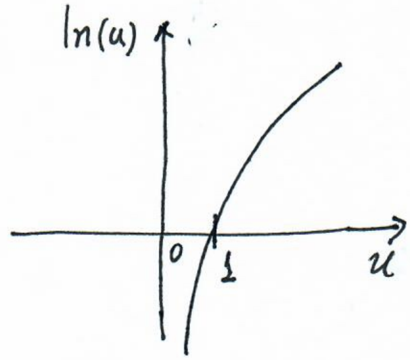
Range of $f = ?$, Domain $= ?$

Since $1+x^2+y^2 > 0$ for all $(x,y) \in \mathbb{R}^2$

and $\ln(u)$ is defined for all $u > 0$

then, Domain $f = \mathbb{R}^2$

Range of f The function $\ln(u)$ has the graph



Clearly, range of $\ln u$ is $\mathbb{R} = (-\infty, \infty)$.

Notice that $1+x^2+y^2 \geq 1$, $(x,y) \in \mathbb{R}^2$

and $\ln(u)$ for $u \geq 1$ is such that $\ln u \in [0, +\infty)$

So range of $f(x,y) = \ln(1+x^2+y^2)$ is $[0, +\infty)$.

Level Curves:

$$f(x,y) = c, \qquad \text{for } c \in (\text{range of } f.)$$

Then for $c \geq 0$

$$\ln(x^2+y^2+1) = c$$

What curves are these?

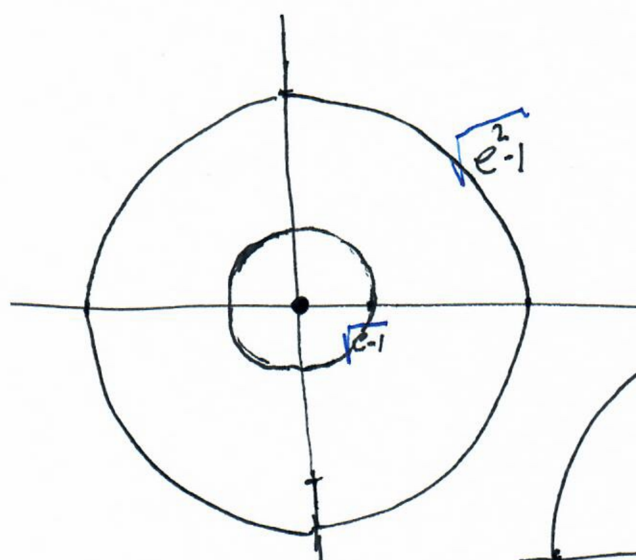
Hard to know!

However,

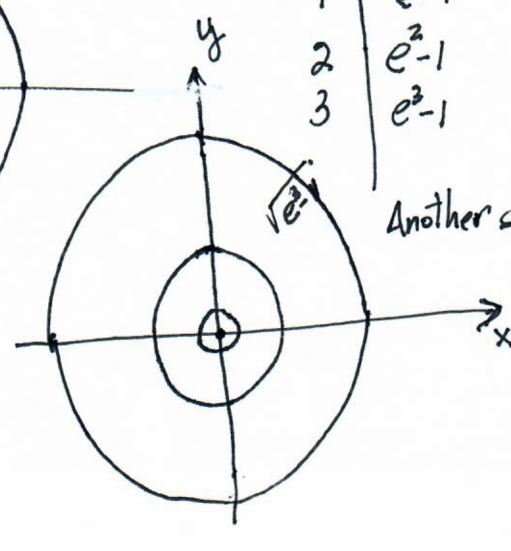
$$e^{\ln(x^2+y^2+1)} = e^c$$

$$\Leftrightarrow \qquad x^2+y^2+1 = e^c \Leftrightarrow \boxed{x^2+y^2 = e^c-1}$$

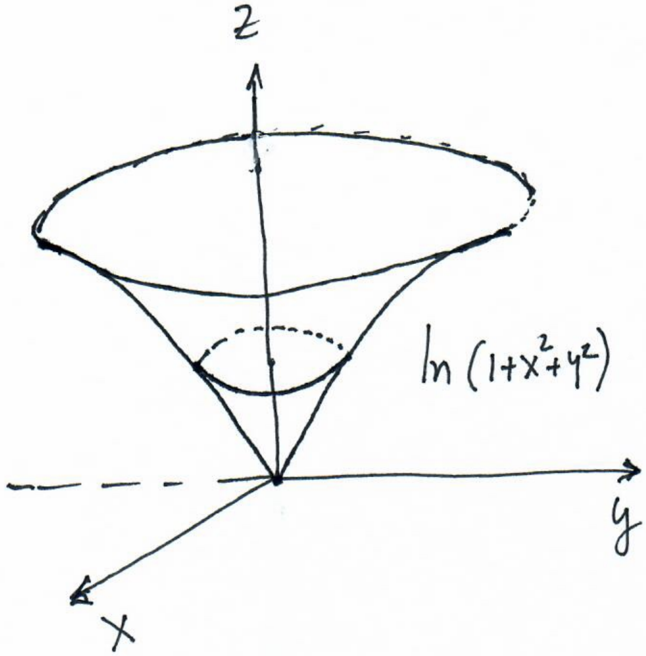
For $c>0$, these are circles of radius e^c-1.

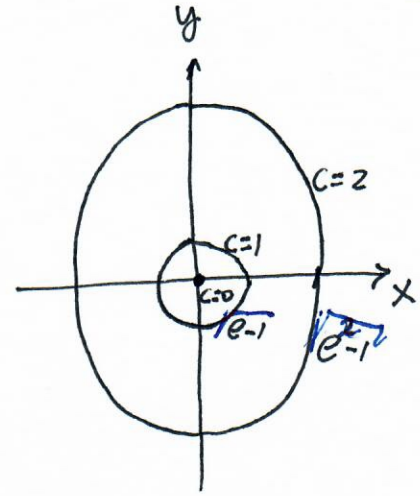


c	e^c-1
0	0
1	$e-1$
2	e^2-1
3	e^3-1



Another scale.


z
$\ln(1+x^2+y^2)$
y
x


y
x
c=2
c=1
c=0
$\sqrt{e-1}$
$\sqrt{e^2-1}$

<u>Exercise</u> $f(x,y) = \dfrac{3y}{x^2+y^2+1}$ Dom f = ?

At the end of Construction of level Curves. SHOW MAPLE!

Find level curves:

$$\frac{3y}{x^2+y^2+1} = c \Rightarrow \frac{3y}{c} = x^2+y^2+1$$

or $x^2 + y^2 - \frac{3}{c}y + 1 = 0$

$$x^2 + \left(y - \frac{3}{2c}\right)^2 + 1 - \frac{9}{4c^2} = 0$$

or $\boxed{x^2 + \left(y - \frac{3}{2c}\right)^2 = \frac{9}{4c^2} - 1}$

$C = 1/2: \quad x^2 + (y-3)^2 = 9 - 1 = 8 = (2\sqrt{2})^2 \simeq (2.82)^2$

$C = 1: \quad x^2 + \left(y - \frac{3}{2}\right)^2 = \frac{9}{4} - 1 = \frac{5}{4} = \left(\frac{\sqrt{5}}{2}\right)^2 \simeq (1.58)^2$

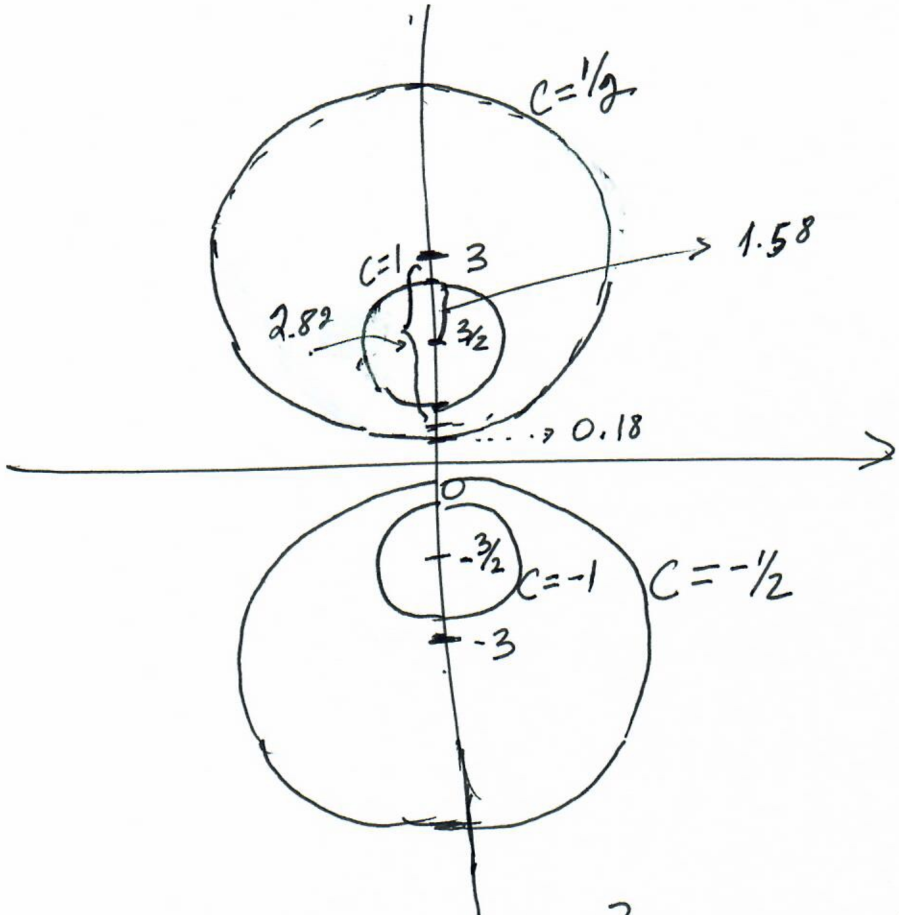
$C = 2: \quad x^2 + \left(y - \frac{3}{4}\right)^2 = \frac{9}{16} - 1 < 0$ No intersection.

$C = 3/2: \quad x^2 + (y-1)^2 = 1 - 1 = 0,$

It implies $(0, 1, 3/2)$ is in graph

and $z = 3/2$ is max. of $f(x,y)$.

It's reached at $(x,y) = (0,1)$.



$C=-1/2: \quad x^2+(y+3)^2 = 9-1=8. = (2\sqrt{2})^2 \simeq (2.82)^2$

$C=1: \quad x^2+(y+\frac{3}{2})^2 = \frac{9}{4}-1=\frac{5}{4}=\left(\frac{\sqrt{5}}{2}\right)^2 \simeq (1.58)^2$

$C=-2$: No intersection.

Show MAPLE Now!

$C=-3/2: x^2+(y+\ \ )^2 = 1-1=0$

Therefore, $z=-3/2$ is min. of $f(x,y)$

It's reached at $(0,-1)$.

range of $f=[-3/2, 3/2]$.

14.1

IN Hwk

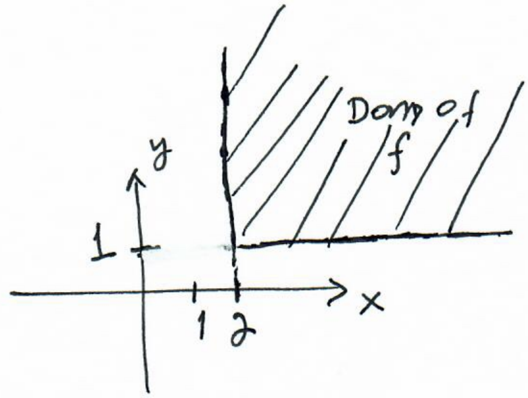
#13) $f(x,y) = \sqrt{x-2} + \sqrt{y-1}$

Domain of f = ?

All (x,y) in $\mathbb{R}^2$ such that

$x-2 \geq 0, \quad y-1 \geq 0.$

$x \geq 2, \quad y \geq 1$



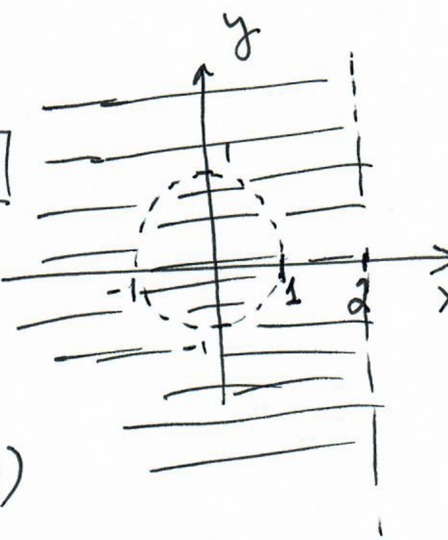
Range of $f = \mathbb{R}^+ \cup \{0\}$

Not in Hwk

#18) $g(x,y) = \dfrac{\ln(2-x)}{1-x^2-y^2}$

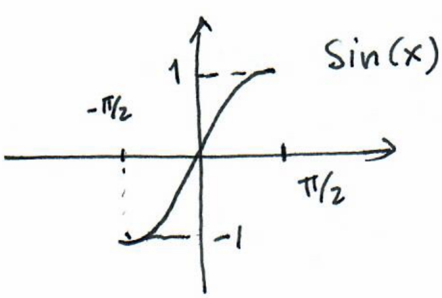
a) $1-x^2-y^2 \neq 0 \Leftrightarrow \boxed{x^2+y^2 \neq 1}$

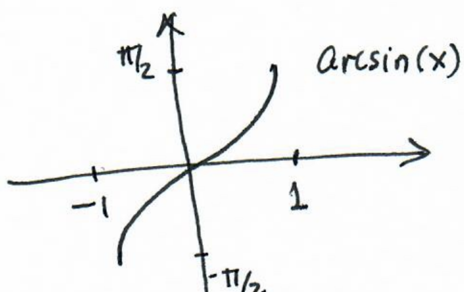
b) $2-x > 0 \Leftrightarrow \boxed{x < 2}$



Not written

#20) $f(x,y) = \sin^{-1}(x+y) = \arcsin(x+y)$





only defined for $x \in [-1,1]$.

Therefore, $\operatorname{dom} f = \{(x,y) \text{ in } \mathbb{R}^2 : -1 < x+y < 1\}$

$\boxed{-1-x < y < 1-x}$

Range $f = [-\pi/2, \pi/2]$